2847. Proposed by B. F. FINKEL, Drury College.

Convert $+ \sqrt{R^2 - x^2}$ into a Fourier series.

2848. Proposed by the late L. G. WELD.

A particle is attracted by a finite, uniform, material right line. Define its path, considering: (a) that the path is the envelope of the instantaneous lines of resultant attraction, as when the particle moves in a highly viscous medium (*i.e.*, without inertia); (b) that the particle moves freely (with inertia).

2849. Proposed by S. A. COREY, Des Moines, Iowa.

In the *Annals of Mathematics*, for April, 1911, Professor Byerly has given a method of approximately representing $f(x)$ in terms of $F_1(x)$, $F_2(x)$, $\cdots$, $F_r(x)$, in an interval $x_0 \leqq x \leqq x_1$. If $F_n(x) = x^n$ and if the mth derivative of $f(x)$ is zero when $x = 0$ for $m > r$, it is evident that Byerly's development becomes exact, and, therefore, identical with Maclaurin's development in the interval $0 \leqq x \leqq x_1$, provided $f(0) = 0$. If $f(0) \neq 0$, it is necessary to replace $f(x)$ by $f(x) - f(0)$ in Byerly's development. The coefficient of x^n in this development is D_n/D, where the D_n and D are certain determinants with elements of the type

$$A_n = \frac{x_1^{2n+1}}{2n+1}, \qquad B_{s,t} = \frac{x_1^{s+t+1}}{s+t+1}, \qquad C_n = \int_0^{x_1} f(x)x^n dx.$$

Prove that as r becomes infinite, D_n/D approaches $f^{[n]}(0)/n!$, the corresponding coefficient in the Maclaurin development, whenever $f(x)$ is analytic, and that any value of x within the range of convergence of the Maclaurin development may be substituted for x_1 without altering the value of D_n/D.

SOLUTIONS OF PROBLEMS.

416 (Algebra) [1914, 156; 1919, 312]   Proposed by C. E. FLANAGAN, Wheeling, W. Va.

The sides of a given rectangle are a and b in which a rectangle is to be inscribed one of whose sides is c. Find the other side, using Euler's rule for quartics.

I. SOLUTION AND DISCUSSION BY OTTO DUNKEL, Washington University.

Let the given rectangle be $RSS'R'$ such that $RS = R'S' = a$ and $RR' = SS' = b$, and let $PP'Q'Q$ be a rectangle inscribed in it such that $QP = Q'P' = c$ and $PP' = QQ' = x$. Suppose further that the vertex P lies on RS, the vertex P' on SS' and so on in order. Let $SP = m$ and $SP' = n$; then from the four similar triangles in the figure, pairs of which are equal, we have at once

(1) $$\frac{n}{x} = \frac{a-m}{c}, \qquad \frac{m}{x} = \frac{b-n}{c}, \qquad n^2 + m^2 = x^2,$$

and from the first two equations follow

(2) $$n + m = \frac{(a+b)x}{c+x}, \qquad n - m = \frac{(a-b)x}{c-x},$$
$$n(c^2 - x^2) = (ac - bx)x, \qquad m(c^2 - x^2) = (bc - ax)x.$$

Squaring and adding the equations in the first line of (2) and using the third equation in (1) we obtain

(3) $$2 - \left(\frac{a+b}{c+x}\right)^2 - \left(\frac{a-b}{c-x}\right)^2 = 0.$$

It will be seen that the form of the equation above is convenient for separating the roots. On clearing of fractions it becomes

(3') $$x^4 - (a^2 + b^2 + 2c^2)x^2 + 4abcx - c^2(a^2 + b^2 - c^2) = 0.$$

The other side x of the inscribed rectangle must satisfy this equation, but in order to say that a given root of this equation determines the remaining side of an *inscribed* rectangle it must be shown that this root gives suitable values of m and n for such a rectangle.

From either (1) or (2) it follows that, if $x = c$, $a = b$. The case $a = b$ is easily disposed of, and the following results may be verified by the above equations. If $x \neq c$, $m = n$ and roots are $x_1 = \sqrt{2} \cdot a - c$, $x_2 = - (\sqrt{2} \cdot a + c)$. If $c < \sqrt{2} \cdot a$, x_1 gives the other side of an inscribed rectangle, which becomes a square if $c = a/\sqrt{2}$ and hence $m = n = a/2$. If $c > \sqrt{2} \cdot a$, then m, n and x_1 are negative and there is, of course, no inscribed rectangle, but there is a rectangle satisfying the analytical conditions, P and P' lying, respectively, on the prolongations of RS and of $S'S$. Also x_2 gives the side of a rectangle with P and P' on the prolongations of SR and SS'. For the double root of (3'), $x = c$, it follows that $n + m = a$. For each value of c such that $a \geqq c \geqq a/\sqrt{2}$, neither m nor n is negative and there is an inscribed square. If $c > a$, m and n have opposite signs and the square is not inscribed but the vertices P and P' are on the prolongations of SR and of $S'S$, respectively. The last two figures are the familiar figures in the proofs of the Pythagorean Theorem.

It will now be assumed that $a > b > 0$. If $c^2 \geqq a^2 + b^2$ it will be obvious that there are no inscribed rectangles. The roots of the equation, which are all real, are easily separated by the method given below for the case $c > a$. It is also easy to determine the positions of the four rectangles. In what follows it will be assumed that $c^2 < a^2 + b^2$.

The equation (3') may be solved by any of the usual methods and there is no difficulty in writing out the expression for the roots other than that of the length and complication of the final result, and besides the result would be neither interesting nor useful. Important functions of the coefficients will be given and by the aid of these the roots may be obtained. Using the notation in Burnside and Panton's *Theory of Equations*, Vol. 1, (1904), page 121, we find for the reducing cubic of (3')

(4) $$4\theta^3 - I\theta + J = 0$$

the following values for the quantities involved

(5)
$$H = - \frac{1}{6} (a^2 + b^2 + 2c^2), \qquad I = \frac{1}{12} (a^2 + b^2 - 4c^2)^2,$$

$$J = \frac{1}{216} [(a^2 + b^2 - 4c^2)^3 + 54c^2(a^2 - b^2)^2].$$

The substitution $\theta = t + H$ gives Euler's cubic. The derivative of the left side of (4) vanishes for $\theta = \pm (a^2 + b^2 - 4c^2)/12$ and for these values of θ the left side takes on the extreme values $c^2(a^2 - b^2)^2/4$ and $[(a^2 + b^2 - 4c^2)^3 + 27c^2(a^2 - b^2)^2]/108$ and the product of these values gives $- \Delta/27$, where Δ is the discriminant (it may also be computed from $\Delta = I^3 - 27J^2$). Thus we have

(6) $$\Delta = - \frac{1}{16} c^2(a^2 - b^2)^2[(a^2 + b^2 - 4c^2)^3 + 27c^2(a^2 - b^2)^2].$$

It is now necessary to examine the last factor in Δ. Considering c^2 as the independent variable, it is seen that this factor has the maximum and minimum values $27a^2(a^2 - b^2)^2/2$ and $27b^2(a^2 - b^2)^2/2$ and hence vanishes only once for real values, passing from positive to negative values. Calling this root c_0^2 it is found to have the value

(7) $$c_0^2 = \frac{a^2 + b^2}{4} + \tfrac{3}{8}(a^2 - b^2)^{2/3}[(a + b)^{2/3} + (a - b)^{2/3}] = \tfrac{1}{8}[(a + b)^{2/3} + (a - b)^{2/3}]^3.$$

If c^2 is greater than c_0^2 the roots of (3') are all real or all imaginary, but since the equation has always one positive and one negative root the roots must be all real in this case. If c^2 is less than c_0^2 there are two imaginary roots in addition to the positive and negative root. For the value c_0^2 there are two equal roots.

In order to determine the number of inscribed rectangles for a given c several cases will be considered, the simplest of which seems to be the one for which $c > a$. This case may be treated without the use of the discriminant Δ. Setting $f(x)$ for the left side of (3), we have

(8)
$$f(0) = 2 \left[1 - \frac{a^2 + b^2}{c^2} \right] < 0, \qquad f(b) = 2 - \left(\frac{a + b}{c + b} \right)^2 - \left(\frac{a - b}{c - b} \right) > 0,$$

$$f \left(\frac{bc}{a} \right) = 2 \left[1 - \frac{a^2}{c^2} \right] > 0, \qquad f(c) = - \infty, \qquad f \left(\frac{ac}{b} \right) = 2 \left[1 - \frac{b^2}{c^2} \right] > 0.$$

The inequalities above follow from $c > a > b$ and $a^2 + b^2 > c^2$. Hence the positive roots lie in the intervals $(0, b)$, $(bc/a, c)$, $(c, ac/b)$. Since $f(-c) = -\infty$ there is a negative root smaller than $-c$. Since cx is the area of the rectangle determined, it can be inscribed only in the case of the root in the interval $(0, b)$. Taking m and n as two quantities determined from (2) by this root, we see from (3) that the equation $n^2 + m^2 = x^2$ is satisfied, and, since x is less than both bc/a and ac/b, both m and n are positive as shown by (2). The equations (1) result from (2) and show that m is less than a and that n is less than b. The equations (1) show also that c is the hypotenuse of a right triangle with the sides $a - m$ and $b - n$ and that c is perpendicular to x. This root furnishes then an inscribed rectangle. In the same way it will be seen that the second positive root makes n positive and m negative, also $n > b$ and $-m < n - b$, $-m > n - a$. The third root makes m positive and greater than a, and n negative, also $-n < m - b$, $-n > m - a$. In the figure for the second positive root Q lies on $R'R$ produced, P on RS produced, and P' on SS' produced. Similarly a figure can be drawn for the third positive root. In the case of the negative root m and n are positive and greater than a and b, respectively. Here P and P' lie, respectively, on the prolongations of SR and of SS', Q' and Q on the prolongations of $R'S'$ and of $R'R$.

If $c = a$ one root is b, and it will be found that of the two remaining positive roots one is less than b and gives an inscribed rectangle while the other is greater than a and hence cannot give an inscribed rectangle. These results show that a is greater than c_0, a fact which is easily verified directly.

The remaining cases may first be separated into two groups; I, $b < c < a$; and II, $0 < c \leqq b$. Again representing the left side of (3) by $f(x)$, we have

$$f'(x) = 2\left[\frac{(a+b)^2}{(c+x)^3} - \frac{(a-b)^2}{(c-x)^3}\right], \qquad f''(x) = -6\left[\frac{(a+b)^2}{(c+x)^4} + \frac{(a-b)^2}{(c-x)^4}\right]$$

(9)

$$f'\left(\frac{bc}{a}\right) = -\frac{4a^3b}{c^3(a^2-b^2)}, \qquad f\left(\frac{ab}{c}\right) = 2\frac{c^4 + a^2b^2}{(c^4 - a^2b^2)^2}(c^2 - a^2)(c^2 - b^2).$$

It is seen from (8) that $f(bc/a)$ is negative in both cases, and the value of the first derivative given in (9) shows that $f(x)$ is decreasing at this point, while the form of the second derivative shows that it continues to decrease as long as x is less than c. Hence, if there are positive roots less than c they must be less than bc/a, and the same reasoning used above will show that each such root gives an inscribed rectangle. When x is greater than c, (9) shows that $f'(x)$ is always positive, and hence $f(x)$ increases from $-\infty$ at $x = c$ and crosses the x-axis only once. From (9) it is seen that in case I the crossing point is beyond ab/c, and hence in this case there is no inscribed rectangle for the root. In case II, (9) shows that the root is less than or equal to ab/c. From (8) it will be seen that this root is greater than or equal to ac/b, since $f(ac/b)$ is negative or zero. It follows that it is also greater than bc/a, and the equations (2) will show that it gives an inscribed rectangle. This disposes of the root greater than c. The examination of the other two roots requires a consideration of the ratio of the sides a and b. An examination of (6) shows that; 1, if $a^2/b^2 \leqq 6\sqrt{3} - 9$, c_0 will lie in II; 2, if $a^2/b^2 > 6\sqrt{3} - 9$, c_0 will lie in I. Hence for case I1, we have $c > c_0$ and therefore two more positive roots. These give inscribed rectangles and they are the only two in this case. For I2, we have two inscribed rectangles if $c > c_0$; one, if $c = c_0$; none if $c < c_0$. Passing now to III1 it follows that if $c > c_0$ there are in all three inscribed rectangles; if $c = c_0$, two; if $c < c_0$, only one. Finally for II2 there is no other inscribed rectangle than the one mentioned above. The roots not giving inscribed rectangles may be interpreted in the same way as in the first part of this discussion.

The character of the roots was obtained at the beginning of this discussion by an examination of the discriminant of the quartic (3') since the wording of the proposed problem seemed to indicate that a discussion of this equation was desirable with reference to Euler's solution of the quartic. But it will be found that that procedure is very tedious as compared with the study of (3), and besides the study of this latter equation does not require any knowledge of the theory of equations. Thus from (9) it will be seen that $f'(x)$ vanishes only once for real values of x and hence there will be two more real roots (in addition to the two always present) or not according as the value of $f(x)$ at this vanishing point is positive or negative. The value of this maximum is easily found, but it is well to find first the value of c_0 for the double root. This is easily found by eliminating x from $f(x) = 0$ and $f'(x) = 0$ and it follows that

(10) $$c_0 = \frac{1}{2\sqrt{2}}[(a+b)^{2/3} + (a-b)^{2/3}]^{3/2},$$

which is easily verified to agree with the previous result. The corresponding double root is then found to be

$$(11) \qquad x_0 = \frac{c_0^{1/3}}{2}\left[(a+b)^{2/3} - (a-b)^{2/3}\right] = \sqrt{\frac{a^2 + b^2 - c_0^2}{3}} = \frac{6abc_0}{a^2 + b^2 + 8c_0^2}.$$

The remaining two roots are obtained by adding to the negative of the double root

$$(12) \qquad \pm\sqrt{\frac{a^2 + b^2 + 8c_0^2}{3}}.$$

The maximum value of $f(x)$ at $x = (x_0/c_0)c$ will be found to be

$$(13) \qquad 2\left[1 - \frac{c_0^2}{c^2}\right],$$

and it gives the same results as the discriminant and in a much more evident form. This shows that the first two positive roots are separated by $(x_0/c_0)c$.

In Osgood's *Calculus*, page 404, is given a solution of the case $a = 15$, $b = 10$, $c = 1$. Here $c_0 = 13.741$, $x_0 = 6.738$, and since $a^2/b^2 = 2.25 > 6\sqrt{3} - 9$ this example falls in II2 and there is only one inscribed rectangle. If $c = 13$ there is no inscribed rectangle. If $c = 14$ there are two inscribed rectangles, one having the side 5.+ and the other the side 8.+.

II. Note on the Preceding by H. P. Manning, Brown University.

There is no particular distinction between the sides c and x of the inscribed rectangle. We can call them x and y, and equation (3) or (3′) will then be represented by a curve of the fourth degree in which many of the results of the above discussion appear graphically.

Moreover, m and n satisfy the equation $m^2 - n^2 = am - bn$, and so are represented by the points of an equilateral hyperbola, which passes through the vertices of the given rectangle if we lay off m and n from its lower left-hand corner.

273 (Number Theory) [1917, 427]. Proposed by V. M. Spunar, Chicago, Ill.

The ratio of the chances that all numbers ending in 1 or 9 and those ending in 3 or 7 are composite is 3 : 2[1].

Note by Norman Anning, University of Maine.

Since

$$1 \times 1 \equiv 3 \times 7 \equiv 9 \times 9 \equiv 1 \pmod{10}, \qquad 1 \times 3 \equiv 7 \times 9 \qquad\qquad \equiv 3 \pmod{10},$$
$$1 \times 7 \equiv 3 \times 9 \qquad\qquad \equiv 7 \pmod{10}, \qquad 1 \times 9 \equiv 3 \times 3 \equiv 7 \times 7 \equiv 9 \pmod{10},$$

the conclusion might be drawn that in the long run as many primes end in 1 as in 9 and as many end in 3 as in 7 and that there would be more of the latter than of the former. A census of primes taken over a considerable range supports these statements but does not point towards the ratio "3 : 2." Counting cases up to 3,200, a number chosen at random, shows the following results:

$$\text{110 primes end in 1, 113 in 3, 116 in 7 and 111 in 9.}$$

Since these numbers are so nearly in equilibrium and since primes are so perfectly lawless no statement could be hazarded about the distribution in a larger interval.

2728 [1918, 397]. Proposed by Norman Anning, University of Maine.

A material triangle of uniform density and thickness is of such a shape that when suspended from the vertices in succession, the lower sides have slopes of 1 : 1, $1\frac{1}{2}$: 1, and 3 : 1. Construct the triangle given that the shortest side is 10 inches.

By definition, an $a : 1$ slope makes an angle with the vertical whose tangent is a.

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[1] The enunciation of the problem is not clear. The chance that all numbers ending in 1 or 9 are composite numbers is zero. — Editors.